

# Topological stability of continuous functions with respect to averagings

Sergiy Maksymenko, Oksana Marunkevych

**ABSTRACT.** We present sufficient conditions for topological stability of continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  having finitely many local extrema with respect to averagings by discrete measures with finite supports.

## 1. Introduction

In applied problems of signals processing, images restoring and digitizing, noise removing, etc., a crucial role is played by linear filters. If  $x(t)$  is a signal, then the result of the application of a linear filter with impulse response  $h(t)$  on time interval  $[0, T]$  is a convolution  $x * h$  of these functions, i.e. a signal defined by the following formula:

$$x * h(t) = \int_0^T x(t - \tau)h(\tau)d\tau.$$

In the case when the support of  $h$  is sufficiently small and  $\int_0^T h(\tau)d\tau = 1$ , the function  $h$  can be regarded as a density of some measure, while the convolution  $x * h$  can be viewed as an *averaging* of  $x$  with respect to this measure. Such averagings are widely used in applications, see e.g. [1], [6], [7].

Notice that a priori the “form” of a averaged signal  $x * h$  can be essentially different of the form of the initial signal  $x$ . For instance, if  $x$  has a unique maximum point, then  $x * h$  may have many maximums. “Preserving form” is a principal requirement to filters in the problems of noise removing, computing entropies of time series, and others, see e.g. [5], [2] and references in these papers.

From mathematical point of view «similarity of forms» of signals means that they are *topologically equivalent* as functions of time, see Definitions 3.1 and 3.4 below.

In the present paper we give wide sufficient conditions for topological stability of averagings of piece wise differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  having finitely many local extrema with respect to discrete measures with finite supports, see Theorems 3.10 and 5.1.

---

2000 *Mathematics Subject Classification.* 26A99, 60G35.

*Key words and phrases.* averaging, topological equivalence.

Those conditions guarantee that after applying to a signal  $x(t)$  a linear filter with impulse response  $h(t)$  being a sum of finitely many  $\delta$ -functions the form of the resulting signal  $x * h$  will not change.

## 2. Averagings of a function

Let  $\mu$  be any probability measure on the closed segment  $[-1, 1]$ . This means that  $\mu$  a non-negative  $\sigma$ -additive measure defined on the Borel algebra of subsets of  $[-1, 1]$  and such that  $\mu[-1, 1] = 1$ . Then for each measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a positive number  $\alpha > 0$  one can define the function  $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$  by the following formula:

$$f_\alpha(x) = \int_{-1}^1 f(x - t\alpha) d\mu. \quad (2.1)$$

We will call  $f_\alpha$  an  $\alpha$ -averaging of  $f$  with respect to the measure  $\mu$ .

Notice that if  $f$  is defined only on some interval  $(a, b)$  and  $2\alpha < b - a$ , then the formula (2.1) determines a function  $f_\alpha$  on the interval  $(a + \alpha, b - \alpha)$ . Moreover,

$$\inf_{y \in [x - \alpha, x + \alpha]} f(y) \leq f_\alpha(x) \leq \sup_{y \in [x - \alpha, x + \alpha]} f(y). \quad (2.2)$$

Consider few simple cases.

1) Suppose  $\mu$  is a discrete measure with finite support. This means that there exists a finite increasing sequence of points  $t_k < t_{k-1} < \dots < t_2 < t_1 \in [-1, 1]$  such that for arbitrary Borel set  $A \subset [-1, 1]$  its measure is given by

$$\mu(A) = \sum_{t_i \in A} \mu(t_i).$$

Then the formula for  $\alpha$ -averaging of  $f : \mathbb{R} \rightarrow \mathbb{R}$  can be represented as follows:

$$f_\alpha(x) = \sum_{i=1}^k f(x - t_i \alpha) \mu(t_i). \quad (2.3)$$

In particular, if  $k = 2$ ,  $t_1 = -1$ ,  $t_2 = +1$  and  $\mu(-1) = \mu(1) = \frac{1}{2}$ , then

$$f_\alpha(x) = \frac{f(x + \alpha) + f(x - \alpha)}{2}. \quad (2.4)$$

2) Suppose that  $\mu$  is absolutely continuous, so there exists a measurable function  $p : [-1, 1] \rightarrow [0, \infty)$  such that  $\mu(A) = \int_A p(t) dt$ . Then

$$f_\alpha(x) = \int_{-1}^1 f(x - t\alpha) p(t) dt.$$

LEMMA 2.1. *The correspondence  $f \mapsto f_\alpha$  is a linear operator on the space of all continuous functions  $C(\mathbb{R}, \mathbb{R})$ . Suppose that  $f \in C(\mathbb{R}, \mathbb{R})$  has one of the following properties:  $f$  is positive, non-negative, negative, non-positive, (strictly) increase, (strictly) decrease, (strictly) convex, (strictly) concave. Then the same property has the  $\alpha$ -averaging  $f_\alpha$  for each  $\alpha > 0$ .*

PROOF. We consider only the cases of (strictly) increasing and convex functions. All other statements are either obvious or can be proved in a similar way and we leave them for the reader.

1) Suppose  $f$  increases, so for all  $x < y \in \mathbb{R}$ ,  $t \in [-1, 1]$ , and  $\alpha > 0$  we have that  $f(x - t\alpha) \leq f(y - t\alpha)$ . Therefore

$$f_\alpha(x) = \int_{-1}^1 f(x - t\alpha) d\mu \leq \int_{-1}^1 f(y - t\alpha) d\mu = f_\alpha(y),$$

that is  $f_\alpha$  also increases.

2) If  $f$  strictly increases, that is  $q(t) = f(y - t\alpha) - f(x - t\alpha) > 0$  for all  $t \in [-1, 1]$ , then

$$f_\alpha(y) - f_\alpha(x) = \int_{-1}^1 q(t) d\mu > 0,$$

since the measure  $\mu$  is non-negative. Hence  $f_\alpha$  is also strictly increasing.

3) Suppose that  $f$  is convex, that is for all  $x, y \in \mathbb{R}$  and  $s \in [0, 1]$  we have that

$$f(sx + (1 - s)y) \leq sf(x) + (1 - s)f(y).$$

Then

$$\begin{aligned} f_\alpha(sx + (1 - s)y) &= \int_{-1}^1 f(sx + (1 - s)y - t\alpha) d\mu \\ &\leq s \int_{-1}^1 f(x - t\alpha) d\mu + (1 - s) \int_{-1}^1 f(y - t\alpha) d\mu \\ &= sf_\alpha(x) + (1 - s)f_\alpha(y). \end{aligned}$$

Lemma is completed.  $\square$

LEMMA 2.2. Let  $f : (a, +\infty) \rightarrow \mathbb{R}$  be a continuous and strictly monotone function. Then  $\lim_{x \rightarrow +\infty} f_\alpha(x) = \lim_{x \rightarrow +\infty} f(x)$  for all  $\alpha > 0$ .

PROOF. For definiteness assume that  $f$  strictly increases. Then it follows from formula (2.2) that for  $x > \alpha$  the following inequalities hold:

$$f(x - \alpha) = \inf_{y \in [x - \alpha, x + \alpha]} f(y) \leq f_\alpha(x) \leq \sup_{y \in [x - \alpha, x + \alpha]} f(y) = f(x + \alpha),$$

whence  $\lim_{x \rightarrow +\infty} f_\alpha(x) = \lim_{x \rightarrow +\infty} f(x)$ .  $\square$

### 3. Topological equivalence of functions

At first we will recall the notion of a *germ* of a function. Let  $a \in \mathbb{R}$ ,  $U$  be a neighborhood of  $a$ , and  $f, g : U \rightarrow \mathbb{R}$  be two continuous functions. Then  $f$  and  $g$  determine the same *germ* at  $a$  whenever  $f = g$  on some neighborhood  $V \subset U$  of  $a$ . The relation “define the same germ at  $a$ ” is obviously an equivalence, and the corresponding equivalence classes are called *germs* at  $a$ . We will denote the class of  $f : U \rightarrow \mathbb{R}$  at  $a$  by  $f : (\mathbb{R}, a) \rightarrow \mathbb{R}$  or by  $f : (\mathbb{R}, a) \rightarrow (\mathbb{R}, f(a))$  if we want to specify the value of  $f$  at  $a$ .

Recall also that a homeomorphism  $\phi : (a, b) \rightarrow (c, d)$  is the same as a continuous surjective strictly monotone function. Moreover, if  $\phi$  increases (decreases) then  $\phi$  is said to *preserve (reverse) orientation*.

DEFINITION 3.1. Let  $a, b \in \mathbb{R}$  and  $f : (\mathbb{R}, a) \rightarrow \mathbb{R}$  and  $g : (\mathbb{R}, b) \rightarrow \mathbb{R}$  be two germs of continuous functions at  $a$  and  $b$  respectively. Then  $f$  and  $g$  are called **topologically equivalent** if there exist two germs of orientation preserving homeomorphisms  $h : (\mathbb{R}, a) \rightarrow (\mathbb{R}, b)$  and  $\phi : (\mathbb{R}, f(a)) \rightarrow (\mathbb{R}, g(b))$  such that  $\phi \circ f = g \circ h$ .

REMARK 3.2. In the definition of topological equivalence it is not necessary to assume that  $\phi$  and  $h$  preserve orientation. However in the present paper we will always do this.

The following simple lemma is left for the reader.

LEMMA 3.3. Let  $f : (\mathbb{R}, a) \rightarrow \mathbb{R}$  and  $g : (\mathbb{R}, b) \rightarrow \mathbb{R}$  be two germs of continuous functions. Suppose also that one of the following conditions holds true:

- (1)  $f$  and  $g$  are strictly monotone on some neighborhoods of  $a$  and  $b$  respectively;
- (2) the points  $a$  and  $b$  are isolated local maximums (resp. local minimums) of  $f$  and  $g$  respectively.

Then  $f$  and  $g$  are topologically equivalent.

DEFINITION 3.4. Two continuous functions  $f : (a, b) \rightarrow \mathbb{R}$  and  $g : (c, d) \rightarrow \mathbb{R}$  are called **topologically equivalent** if there exist orientation preserving homeomorphisms

$$h : (a, b) \rightarrow (c, d), \quad \phi : \mathbb{R} \rightarrow \mathbb{R},$$

such that  $\phi \circ f = g \circ h$ , that is they made commutative the following diagram:

$$\begin{array}{ccc} (a, b) & \xrightarrow{f} & \mathbb{R} \\ h \downarrow & & \downarrow \phi \\ (c, d) & \xrightarrow{g} & \mathbb{R} \end{array}$$

We will now recall some results about classification of continuous functions on the real line up to a topological equivalence.

DEFINITION 3.5. [3] A **generalized snake** of length  $k$ , or simply a  **$k$ -snake** is an arbitrary sequence of  $k$  numbers  $\{A_1, \dots, A_k\}$ . Two  $k$ -snakes  $\{A_1, \dots, A_k\}$  and  $\{B_1, \dots, B_k\}$  are **equivalent** whenever for any  $i, j = 1, \dots, k$  the following condition holds true:

$$(*) \quad A_i < A_j \text{ if and only if } B_i < B_j;$$

Evidently, this condition also implies that  $A_i = A_j$  if and only if  $B_i = B_j$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function having only finitely many local extremes  $x_1, \dots, x_n$  and being strictly monotone on the complementary intervals to these points. In particular, it follows that there exist finite or infinite limits  $A_0 = \lim_{x \rightarrow -\infty} f(x)$  and  $A_{n+1} = \lim_{x \rightarrow +\infty} f(x)$ . Denote  $A_i = f(x_i)$ ,  $i = 1, \dots, n$ . Then the sequence of numbers  $\xi(f) = \{A_0, \dots, A_{n+1}\}$  will be called a *snake associated with  $f$* .

The following statement is well-known and can be easily proved.

LEMMA 3.6. e.g. [8], [3] *Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions both having exactly  $k$  local extremes for some  $k \geq 0$  and being strictly monotone on the complementary intervals to these points. Then  $f$  and  $g$  are topologically equivalent if and only if the corresponding snakes  $\xi(f)$  and  $\xi(g)$  are equivalent.*  $\square$

DEFINITION 3.7. *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function and  $\mu$  be a probability measure on  $[-1, 1]$ . We will say that  $f$  is **topologically stable** with respect to the averagings by measure  $\mu$  whenever there exists  $\varepsilon > 0$  such that for all  $\alpha \in (0, \varepsilon)$  the functions  $f$  and  $f_\alpha$  are topologically equivalent.*

Similarly, one can give a definition of a local topological stability of averagings by measure  $\mu$ . Let  $f : (\mathbb{R}, a) \rightarrow \mathbb{R}$  be a germ of a continuous function at a point  $a \in \mathbb{R}$ . This means that  $f$  is a continuous function defined on the interval  $(a - \varepsilon, a + \varepsilon)$  for some  $\varepsilon > 0$ . Then it follows from (2.1) that for  $\alpha < \varepsilon/2$  the averaging  $f_\alpha$  is correctly defined on the interval  $(a - \varepsilon/2, a + \varepsilon/2)$ . Moreover, the germ of  $f_\alpha$  at  $a$ , evidently, depends only on the germ of  $f$  at that point.

REMARK 3.8. Notice the germs  $f$  and  $f_\alpha$  at  $a$  are in general not topologically equivalent. For example, if  $a$  is an isolated local minimum of  $f$ , then  $f_\alpha$  may also have an isolated local minimum  $b$  very closed to  $a$  but distinct from  $a$ . Then the germs of  $f$  and  $f_\alpha$  at  $a$  are not topological equivalent, though by Lemma 3.3 the restriction of  $f$  on some neighborhood  $(c_1, c_2)$  of  $a$  will be topologically equivalent to the restriction of  $f_\alpha$  to some neighborhood  $(d_1, d_2)$  of  $b$ . This observation leads to the following definition.

DEFINITION 3.9. *A germ  $f : (\mathbb{R}, a) \rightarrow \mathbb{R}$  is said to be **topologically stable** with respect to averagings by measure  $\mu$  if there exists  $\varepsilon > 0$  such that for each  $\alpha \in (0, \varepsilon)$  the following condition holds true:*

*there exist  $c_1, c_2, d_1, d_2 \in (a - \varepsilon, a + \varepsilon)$  depending on  $\alpha$  and such that  $c_1 < a < c_2$ ,  $d_1 < d_2$ , and the restrictions*

$$f|_{(c_1, c_2)} : (c_1, c_2) \rightarrow \mathbb{R}, \quad f_\alpha|_{(d_1, d_2)} : (d_1, d_2) \rightarrow \mathbb{R}$$

*are topologically equivalent.*

In the present paper we give sufficient conditions for topological stability of averagings of piece-wise differentiable functions with respect to averaging by discrete probability measures with finite supports.

The following theorem shows that for functions of “general position” with finitely many local extremes a local stability with respect to averagings by measure  $\mu$  implies global stability.

THEOREM 3.10. *Let  $\mu$  be a probability measure on  $[-1, 1]$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function having only finitely many local extremes  $x_1, \dots, x_n$  and being strictly monotone on the complement to these points. As above denote*

$$A_0 = \lim_{x \rightarrow -\infty} f(x), \quad A_i = f(x_i), \quad i = 1, \dots, n, \quad A_{n+1} = \lim_{x \rightarrow +\infty} f(x).$$

*Suppose that the following conditions hold:*

- (1) *the numbers  $A_1, \dots, A_n$  are mutually distinct and differ from  $A_0$  and  $A_{n+1}$  as well;*

- (2) for each  $i = 1, \dots, n$  the germ  $f : (\mathbb{R}, x_i) \rightarrow \mathbb{R}$  of  $f$  at  $x_i$  is topologically stable with respect to averagings by  $\mu$ .

Then  $f$  is topologically stable with respect to averagings by  $\mu$ .

PROOF. It suffices to find  $\varepsilon > 0$  such that the snakes  $\xi(f)$  and  $\xi(f_\alpha)$  are equivalent for all  $\alpha \in (0, \varepsilon)$ . Then it will follow from Lemma 3.6 that  $f$  and  $f_\alpha$  are topologically equivalent, whence  $f$  will be topologically stable with respect to averagings by  $\mu$ .

Since  $f$  has only finitely many local extremes, it follows from (2) and Lemma 2.1 that there exists  $\varepsilon > 0$  such that for all  $\alpha \in (0, \varepsilon)$  the averaging  $f_\alpha$  has also exactly  $n$  local extremes. Let  $\xi(f_\alpha) = \{B_0, B_1, \dots, B_{n+1}\}$  be the corresponding snake for  $f_\alpha$ .

Then by Lemma 2.2  $A_0 = B_0$  and  $A_{n+1} = B_{n+1}$ . Moreover, it follows from inequalities (2.2) that one can reduce  $\varepsilon$  so that for each pair  $i \neq j$  the condition  $A_i < A_j$  will imply that  $B_i < B_j$  as well and that  $B_i$  also differs from  $B_0$  and  $B_{n+1}$ . This implies that the snakes  $\xi(f)$  and  $\xi(f_\alpha)$  are equivalent.  $\square$

#### 4. Topological stability of germs with respect to averagings

Let  $\varepsilon > 0$  and  $f : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$  be a continuous function such that 0 is an isolated local minimum for  $f$  and that  $f$  monotone decreases on  $(-\varepsilon, 0]$  and monotone increases on  $[0, \varepsilon)$ . It will be convenient to denote

$$f_L = f|_{(-\varepsilon, 0]} : (-\varepsilon, 0] \longrightarrow \mathbb{R}, \quad f_R = f|_{[0, \varepsilon)} : [0, \varepsilon) \longrightarrow \mathbb{R}.$$

LEMMA 4.1. *Let  $\mu$  be a probability measure on  $[-1, 1]$ . Then each of the following conditions implies that the germ of  $f$  at 0 is locally stable with respect to averagings by  $\mu$ :*

- (1)  $f$  is strictly convex;
- (2)  $f$  is  $C^1$ -differentiable  $(-\varepsilon, 0) \cup (0, +\varepsilon)$  and  $f'$  is strictly increasing on  $(-\varepsilon, 0) \cup (0, +\varepsilon)$ ;
- (3)  $f$  is  $C^2$ -differentiable on some neighborhood of 0 and  $f''(0) > 0$ .

PROOF. Evidently, (3)  $\Rightarrow$  (2)  $\Rightarrow$  (1), so it suffices to prove (1).

(1) Suppose  $f$  is strictly convex and let  $\alpha < 2\varepsilon$ . Then by Lemma 2.1 the averaging  $f_\alpha$  is also strictly convex, whence it has a unique minimum point as well as  $f$ . Then by Lemma 3.3  $f$  and  $f_\alpha$  are topological equivalent and so the germ of  $f$  at 0 is topological stable with respect to averagings by measure  $\mu$ .  $\square$

REMARK 4.2. One can assume that in (3) of Lemma 4.1 the homeomorphisms  $h$  and  $\phi$  satisfying  $\phi \circ f = f_\alpha \circ h$  are diffeomorphisms. Indeed, the assumption that  $f$  belongs to class  $C^2$  near 0 and  $f''(0) > 0$  means 0 is a non-degenerate critical point. Moreover, for all small  $\alpha > 0$  the function  $f_\alpha$  will also belong to class  $C^2$  and also will have a unique minimum point, say  $x_\alpha$ , with  $f''_\alpha(x_\alpha) > 0$ . Therefore  $x_\alpha$  is a non-degenerate critical point for  $f_\alpha$  as well. Then by Morse Lemma the germs  $f : (\mathbb{R}, 0) \rightarrow \mathbb{R}$  and  $f_\alpha : (\mathbb{R}, x_\alpha) \rightarrow \mathbb{R}$  are smoothly equivalent, that is  $h$  and  $\phi$  can be chosen to be diffeomorphisms, see [4, Theorem II.6.9, Proposition III.2.2].

## 5. Main result

Let  $\mu$  be a probability measure on  $[-1, 1]$  with finite support  $t_k < t_{k-1} < \dots < t_1$ . Put  $p_i = \mu(t_i)$ ,  $i = 1, \dots, k$ . Then  $p_i > 0$  and  $p_k + \dots + p_1 = 1$ .

In what follows we will assume that the function  $f : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$  belongs to the class  $C^1$  on  $(-\varepsilon, 0) \cup (0, +\varepsilon)$  and there exist finite or infinite limits

$$L = \lim_{x \rightarrow 0-0} f'_L(x), \quad R = \lim_{x \rightarrow 0+0} f'_R(x). \quad (5.5)$$

Evidently,  $L \leq 0$  and  $R \geq 0$ .

If  $L$  and  $R$  are finite, then for each  $j = 1, \dots, k-1$  we introduce the following numbers:

$$X_j = L(p_1 + \dots + p_j) + R(p_{j+1} + \dots + p_k). \quad (5.6)$$

It is easy to see that they satisfy the following inequality:

$$L \leq X_{k-1} < X_{k-2} < \dots < X_1 \leq R.$$

**THEOREM 5.1.** *Suppose that one of the following conditions holds true:*

- (a) *both limits  $L$  and  $R$  are finite and  $X_j \neq 0$  for  $j = 1, \dots, k-1$ .*
- (b) *one of the limits either  $L$  or  $R$  is infinite and the other one is finite.*

*Then the germ of  $f$  at 0 is topologically stable with respect to averagings by measure  $\mu$ .*

In order to make the proof of Theorem 5.1 more clear we will first formulate and prove a special case.

**LEMMA 5.2.** *Let  $\mu$  be a discrete measure on  $[-1, 1]$  such that  $\mu(-1) = \mu(1) = \frac{1}{2}$  and so*

$$f_\alpha(x) = \frac{f(x+\alpha) + f(x-\alpha)}{2} = \frac{1}{2} \cdot \begin{cases} f_L(x+\alpha) + f_L(x-\alpha), & x \in (-\varepsilon + \alpha, -\alpha), \\ f_L(x+\alpha) + f_R(x-\alpha), & x \in [-\alpha, \alpha], \\ f_R(x+\alpha) + f_R(x-\alpha), & x \in (\alpha, \varepsilon - \alpha). \end{cases}$$

*see Eq. (2.4). Suppose that both limits  $L$  and  $R$  are finite and  $L + R \neq 0$ . Then germ of  $f$  at 0 is topologically stable with respect to the averagings by  $\mu$ .*

Notice that under the assumptions on  $\mu$  we have in Lemma 5.2 that  $k = 2$ ,  $p_2 = p_1 = \frac{1}{2}$ . Hence

$$X_1 = \frac{1}{2}(L + R) \neq 0, \quad (5.7)$$

Thus Lemma 5.2 is a particular case of Theorem 5.1.

**PROOF OF LEMMA 5.2.** It suffices to find  $\delta > 0$  such that for all  $\alpha \in (0, \delta/2)$  the function  $f_\alpha$  will have a unique minimum as well as  $f$ . Then by Lemma 3.3  $f$  and  $f_\alpha$  will be topologically equivalent.

Since the limits (5.5) are finite, the derivatives  $f'_L$  and  $f'_R$  continuously extend to the closed segments  $(-\varepsilon, 0]$  and  $[0, \varepsilon)$  respectively, so that  $f'_L(0) = L$  and  $f'_R(0) = R$ .

Then we get from (5.7) and continuity of  $f'_L$  and  $f'_R$  it follows that there exists  $\delta \in (0, \varepsilon)$  such that for any  $x, y \in (0, \delta)$  the following inequality holds:

$$f'_L(-x) + f'_R(y) \neq 0.$$

By assumption  $f_L$  is monotone decreasing on  $(-\varepsilon, 0)$  while  $f_R$  is monotone increasing on  $(0, \varepsilon)$ . Hence we get from Lemma 2.1 that  $f_\alpha$  is monotone increasing on  $(-\varepsilon + \alpha, -\alpha)$  and monotone decreasing on  $(\alpha, \varepsilon - \alpha)$ . We will show that  $f_\alpha$  is strictly monotone on  $(-\alpha, \alpha)$ . Therefore  $f_\alpha$  will have a minimum point at one of the ends of the segment  $[-\alpha, \alpha]$  in accordance with the sign of  $L + R$ .

For definiteness assume that  $L + R > 0$ , whence  $f'_L(-x) + f'_R(y) > 0$  for all  $x, y \in (0, \delta)$ . Therefore for  $\alpha \in (0, \delta/2)$  and  $x \in (-\alpha, \alpha) \subset (-\delta/2, \delta/2)$  we have that

$$-\delta < x - \alpha < 0 < x + \alpha < \delta,$$

whence

$$f'_\alpha(x) = f'_L(x + \alpha) + f'_R(x - \alpha) > 0.$$

Thus  $f_\alpha$  is strictly monotone on  $[-\alpha, \alpha]$ . On the other hand,  $f_\alpha$  is strictly increasing on  $(-\varepsilon + \alpha, -\alpha]$  and strictly increasing on  $[\alpha, \varepsilon - \alpha]$ . Therefore  $f_\alpha$  has a unique minimum point  $x = -\alpha$ .  $\square$

Let us also show that the assumption (5.7) is essential.

COUNTEREXAMPLE 5.3. Let  $f(x) = |x|$ . Then  $f_L(x) = -x$ ,  $f_R(x) = x$ ,  $L = f'_L(0) = -1$  and  $R = f'_R(0) = +1$ , whence

$$X_1 = L + R = -1 + 1 = 0,$$

so the condition (5.7) fails. In this case for every  $\alpha > 0$

$$f_\alpha(x) = \frac{1}{2}(|x + \alpha| + |x - \alpha|) = \begin{cases} -x, & x \in (-\infty, -\alpha), \\ \alpha, & x \in [-\alpha, \alpha], \\ x, & x \in (\alpha, +\infty). \end{cases}$$

Thus  $f_\alpha$  is constant on the segment  $[-\alpha, \alpha]$ , whence it is not topologically equivalent to  $f$ .

## 6. Proof of Theorem 5.1

If  $k = 1$ , then  $f_\alpha(x) = f(x - t_1\alpha)$ , whence  $f$  and  $f_\alpha$  are topologically equivalent.

So assume that  $k \geq 2$ . It suffices to show that there exists  $\delta > 0$ , such that for  $\alpha \in (0, \delta/2)$  the function  $f_\alpha$  has a unique minimum point as well as  $f$ . Then by Lemma 3.3  $f$  and  $f_\alpha$  will be topologically equivalent.



Notice that  $f_\alpha$  is given by the following formulas:

$$f_\alpha(x) = \begin{cases} \sum_{i=1}^k f_L(x - t_i\alpha)p_i, & x \in (-\varepsilon + \alpha, t_k\alpha), \\ \sum_{i=1}^j f_L(x - t_i\alpha)p_i + \sum_{i=j+1}^k f_R(x - t_i\alpha)p_i, & x \in [t_{j+1}\alpha, t_j\alpha), \\ k-1 \geq j \geq 1, \\ \sum_{i=1}^k f_R(x - t_i\alpha)p_i, & x \in [t_1\alpha, \varepsilon - \alpha). \end{cases} \quad (6.8)$$

Indeed, it follows from the condition  $x \in (-\varepsilon + \alpha, t_k\alpha)$  that  $x - t_k\alpha < 0$ , and therefore  $x - t_i\alpha < 0$  for all  $i = 1, \dots, k$ . Hence the value  $f_\alpha(x)$  is given by the first line of formula (6.8).

Further, the assumption  $x \in [t_{j+1}\alpha, t_j\alpha)$  is equivalent to  $x - t_j\alpha < 0 \leq x - t_{j+1}\alpha$ , whence

$$x - t_1\alpha < \dots < x - t_j\alpha < 0 \leq x - t_{j+1}\alpha < \dots < x - t_k\alpha.$$

Therefore  $f_\alpha(x)$  is given by the second line of formula (6.8).

Similarly, it follows from the assumption  $x \in [t_1\alpha, \varepsilon - \alpha)$  that  $x - t_i\alpha \geq 0$  for all  $i = 1, \dots, k$ , and therefore the value  $f_\alpha(x)$  is given by the third line of (6.8).

By assumption  $f_L$  is monotone decreasing on  $(-\varepsilon, 0)$ , while  $f_R$  is monotone decreasing on  $(0, \varepsilon)$ . Then by Lemma 2.1  $f_\alpha$  is monotone decreasing on  $(-\varepsilon + \alpha, t_k\alpha)$  and monotone increasing on  $(t_1\alpha, \varepsilon - \alpha)$ . We will show that for some  $m \in \{1, \dots, k\}$  the function  $f_\alpha$  is strictly decreasing on  $(t_k\alpha, t_m\alpha)$  and strictly increasing on  $(t_m\alpha, t_1\alpha)$ . This will imply that  $f_\alpha$  has a unique minimum point  $t_m\alpha$ .

For each  $j = k-1, \dots, 2, 1$  define a function  $g_j : (0, \varepsilon)^k \rightarrow \mathbb{R}$  by the following formula:

$$g_j(x_1, \dots, x_k) = \sum_{i=1}^j f'_L(-x_i)p_i + \sum_{i=j+1}^k f'_R(x_i)p_i.$$

LEMMA 6.1. *Suppose that there exist  $\delta \in (0, \varepsilon)$  and  $m \in \{1, \dots, k\}$  such that for all  $(x_1, \dots, x_k) \in (0, \delta)^k$  the following inequalities hold:*

$$\begin{aligned} g_j(x_1, \dots, x_k) &< 0, & j \geq m, \\ g_j(x_1, \dots, x_k) &> 0, & j < m. \end{aligned} \quad (6.9)$$

*Then for each  $\alpha \in (0, \delta/2)$  the function  $f_\alpha$  has a unique minimum point  $x = t_m\alpha$ .*

PROOF. Let  $\alpha \in (0, \delta/2)$ ,  $j \in \{k-1, \dots, 2, 1\}$  and  $x \in (t_{j+1}\alpha, t_j\alpha)$ . Then by formula (6.8),

$$\begin{aligned} f'_\alpha(x) &= \sum_{i=1}^j f'_L(x + t_i\alpha)p_i + \sum_{i=j+1}^k f'_R(x + t_i\alpha)p_i \\ &= g_j(-x - t_1\alpha, \dots, -x - t_j\alpha, x + t_{j+1}\alpha, \dots, x + t_k\alpha). \end{aligned}$$

Also notice that  $|x| < \alpha$ , whence

$$|x - t_i\alpha| \leq |x| + \alpha < 2\alpha < \delta, \quad i = 1, \dots, k.$$

Then it follows from (6.9) that  $f'_\alpha(x) < 0$  for  $x < t_m\alpha$  and  $f'_\alpha(x) > 0$  for  $x > t_m\alpha$ .

Thus the derivative  $f'_\alpha$  is defined on  $(t_k\alpha, t_1\alpha)$  except possibly finitely many points of the form  $t_i\alpha$ ,  $i = 1, \dots, k$ , takes negative values on  $(t_k\alpha, t_m\alpha)$  and positive values of  $(t_m\alpha, t_1\alpha)$ . Hence  $f_\alpha$  has a unique minimum point  $x = t_m\alpha$ .  $\square$

It remains to check that each of the conditions (a) and (b) of Theorem 5.1 implies (6.9).

(a) Suppose that the limits  $L$  and  $R$  are finite and  $X_j \neq 0$  for all  $j = 1, \dots, k-1$ . It follows from finiteness of  $L$  and  $R$  that  $f'_L$  and  $f'_R$  extend to  $[0, \varepsilon)$  and  $(-\varepsilon, 0]$  respectively by  $f'_L(0) = L$  and  $f'_R(0) = R$ . Therefore, see (5.6),

$$\begin{aligned} X_j &= L(p_1 + \dots + p_j) + R(p_{j+1} + \dots + p_k) \\ &= \sum_{i=1}^j f'_L(0) p_i + \sum_{i=j+1}^k f'_R(0) p_i = g_j(0, \dots, 0) \neq 0. \end{aligned} \quad (6.10)$$

Hence by continuity of  $f'_L$  and  $f'_R$  there exists  $\delta > 0$  such that for all  $(x_1, \dots, x_k) \in (0, \delta)^k$  and  $j = 1, \dots, k-1$  the value  $g_j(x_1, \dots, x_k)$  has the same sign as  $X_j$ . Due to Lemma 6.1 this sign also coincides with the sign of the derivative  $f'_\alpha$  on the interval  $(t_{j+1}\alpha, t_j\alpha)$ .

Recall that  $L < X_{k-1} < \dots < X_1 < R$  and  $L \leq 0 \leq R$ . Hence there exists  $m \in \{k, k-1, \dots, 1\}$  such  $X_j < 0$  for  $k-1 \geq j \geq m$  and  $X_j > 0$  for  $1 \leq j < m$ . Therefore the assumptions (6.9) of Lemma 6.1 hold, whence  $f$  has a unique minimum at  $t_m\alpha$ . Let us explain this in more details:

- (i) if  $0 < X_{k-1} < \dots < X_1$ , then  $m = k$  and  $f$  decreases on  $(-\varepsilon, t_k\alpha]$  and increases on  $[t_k\alpha, \varepsilon)$ , and so  $f$  has a minimum point  $t_k\alpha$ ;
- (ii) if  $X_{k-1} < \dots < X_m < 0 < X_{m+1} < \dots < X_1$ , then  $f$  decreases on  $(-\varepsilon, t_m\alpha]$  and increases on  $[t_m\alpha, \varepsilon)$ , and so  $f$  has a minimum point  $t_m\alpha$ ;
- (iii) if  $X_{k-1} < \dots < X_1 < 0$ , then  $m = 1$ ,  $f$  decreases on  $(-\varepsilon, t_1\alpha]$  and increases on  $[t_1\alpha, \varepsilon)$ , whence  $f$  has a minimum point  $t_1\alpha$ .

(b) Suppose that  $|L| < \infty$  and  $R = +\infty$ . Then one can find  $\delta > 0$  such that  $g_j(x_1, \dots, x_k) > 0$  for all  $(x_1, \dots, x_k) \in (0, \delta)^k$  and  $j = 1, \dots, k-1$ . This means that the assumptions of Lemma 6.1 hold for  $m = k$ , whence  $f_\alpha$  will have a unique minimum point  $x = t_k\alpha$ .

Similarly if  $L = -\infty$  and  $|R| < \infty$ , then the function  $f_\alpha$  will have a unique minimum point  $x = t_1\alpha$ . Theorem 5.1 is completed.  $\square$

## References

- [1] S. A. Akhmanov, Yu. E. D'yakov, and A. S. Chirkin, *Vvedenie v statisticheskuyu radiofiziku i optiku*, "Nauka", Moscow, 1981. MR 626992(83c:78001)
- [2] Alexandra Antoniouk, Karsten Keller, and Sergiy Maksymenko, *Kolmogorov-Sinai entropy via separation properties of order-generated  $\sigma$ -algebras*, Discrete Contin. Dyn. Syst. **34** (2014), no. 5, 1793–1809. MR 3124713

- [3] V. I. Arnol'd, *Snake calculus and the combinatorics of the Bernoulli, Euler and Springer numbers of Coxeter groups*, Uspekhi Mat. Nauk **47** (1992), no. 1(283), 3–45, 240. MR 1171862 (93h:20042)
- [4] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Springer-Verlag, New York-Heidelberg, 1973, Graduate Texts in Mathematics, Vol. 14. MR 0341518 (49 #6269)
- [5] C. Bandt, B. Pompe, *Permutation entropy: A natural complexity measure for time series*, Phys. Rev. Lett. **88** (2002), 174102.
- [6] Kenneth R. Crounse, *Methods for Image Processing and Pattern Formation in Cellular Neural Networks: A Tutorial*, Transactions on Circuits and Systems-1: fundamental theory and application, IEEE, **42** (1995), no. 10, 583–601.
- [7] P. Milanfar, *A tour of modern image filtering: new insights and methods, both practical and theoretical*, Signal Processing Magazine, IEEE, **30**(1) (2013) 106–128 p.
- [8] René Thom, *L'équivalence d'une fonction différentiable et d'un polynome*, Topology **3** (1965), no. suppl. 2, 297–307. MR 0187249 (32 #4702)

INSTITUTE OF MATHEMATICS OF NAS OF UKRAINE, STR. TERESHCHENKIVS'KA, 3, 01601, KYIV, UKRAINE

*E-mail address:* maks@imath.kiev.ua, oxanamarunkevych@rambler.ru